APPLIED MATHEMATICS FOR ENGINEERS FINAL EXAM		
Code : *MAT 210* Acad.Year : *2018-19* Semester : *Spring*	Last Name : Name :	#:
Date : *24.05.2019*	Student ID : Signature :	
Time : *9:00* Duration : *110 min*	8 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	

P1. (20)	P2. (26)	P3. (18)	P4. (20)	P5. (16)		Total. (100)

1. *(10×2=20pts)* Indicate whether a given statement is **TRUE** or **FALSE** by circling your answer. No explanations are required.

Point values are: Incorrect=0pts, Blank=1pt, Correct=2pts.

TRUE / FALSE The complex Fourier transform is given by $c_k = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\, e^{-i\,kt}\, dt$

TRUE / FALSE The complex inverse Fourier transform is given by $f(t) = \displaystyle\sum_{k=0}^{\infty} c_k\, e^{i\,kt}$

TRUE / FALSE The discrete Fourier transform is given by $c_k = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} f_n\, \omega_N^{-k\,n}$

TRUE / FALSE The discrete inverse Fourier transform is given by $f_n = \displaystyle\sum_{k=0}^{N-1} c_k\, \omega_N^{n\,k}$

TRUE / FALSE For roots of unity, $\overline{\omega_N} = e^{2\pi/N\, i}$ $\left(\overline{\omega_N} = e^{-\frac{2\pi}{N} i}\right)$

TRUE / FALSE For roots of unity, $\overline{\omega_N} = \omega_N^{-1}$

TRUE / FALSE For roots of unity, $\omega_6^{-9} = \overline{\omega_3}$ $\left(\omega_6^{-8} = \overline{\omega_3}\right)$

TRUE / FALSE If f has length 6, then $c_1 = \overline{c_6}$ $\left(c_1 = \overline{c_5}\right)$

TRUE / FALSE For functions $f(t)$, we have $\mathcal{F}_k\{f(t)\} = \mathcal{F}_{2k}\{f(2t)\}$

TRUE / FALSE Convolutions satisfy the formula $\mathcal{F}\{\mathbf{f} \circledast \mathbf{g}\} = N\mathcal{F}\{\mathbf{f}\} \circledast \mathcal{F}\{\mathbf{g}\}$.

$\left(\mathcal{F}\{f \circledast g\} = N\,\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\right)$

2. *(2×4=8pts)* Suppose $f(t) = \cos(3t) + 2\sin(4t)$

(A) Write all non-zero **real** Fourier coefficients of $f(t)$.

$a_3 = 1, \quad b_4 = 2$

(B) Write all non-zero **real** Fourier coefficients of $\int f(t)\,dt$.

$\int f(t)dt = \frac{1}{3}\sin(3t) - \frac{1}{2}\cos(4t) + C$

$a_0 = C, \quad a_4 = -\frac{1}{2}, \quad b_3 = \frac{1}{3}$

3. *(2×4=8pts)* Suppose the only nonzero complex Fourier coefficients of $g(t)$ with positive index are $c_1 = i$ and $c_2 = (1+i)$.

(A) Write the complex exponential formula for $g(t)$.

$c_1 = i \Rightarrow c_{-1} = -i$

$c_2 = 1+i \Rightarrow c_{-2} = 1-i$

So: $g(t) = (1-i)e^{-2it} - ie^{-it} + ie^{it} + (1+i)e^{2it}$

(B) Write the real (sine/cosine) formula for $g(t)$.

$c_1 = i \Rightarrow b_1 = -2\,Im(c_1) = -2$

$c_2 = 1+i \Rightarrow \begin{cases} a_1 = 2\,Re(c_2) = 2 \\ b_2 = -2\,Im(c_2) = -2 \end{cases}$

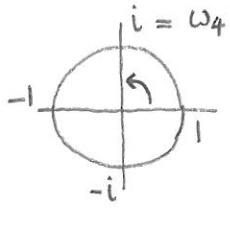
So: $g(t) = 2\cos(2t) - 2\sin(t) - 2\sin(2t)$

4. *(10pts)* Find f so that $\mathcal{F}\{\mathbf{f}\} = \begin{bmatrix} 2 & (1+i) & 0 & (1-i) \end{bmatrix}$.

$\Updownarrow$

$\underline{f} = \mathcal{F}^{-1}[2 \quad (1+i) \quad 0 \quad (1-i)]$

Inverse discrete Fourier with $\omega_4 = i$:

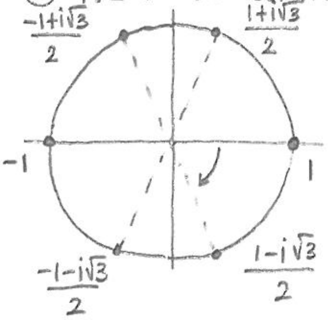


- $f_0 = 2\cdot1 + (1+i)\cdot1 + 0\cdot1 + (1-i)\cdot1 = 4$
- $f_1 = 2\cdot1 + (1+i)\cdot i + 0\cdot -1 + (1-i)\cdot -i = 0$
- $f_2 = 2\cdot1 + (1+i)\cdot -1 + 0\cdot1 + (1-i)\cdot -1 = 0$
- $f_3 = 2\cdot1 + (1+i)\cdot -i + 0\cdot -1 + (1-i)\cdot i = 4$

So: $\underline{f} = [4 \quad 0 \quad 0 \quad 4]$

5. *(18pts)* Compute the discrete Fourier transform of $\mathbf{f} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \end{bmatrix}$

⊛ $N = 6$ so divide $\mathbb{C}$ unit circle into 6 pieces:



⊛ $\mathcal{F}_0\{f\} = \frac{1}{6}(1+2+1+2+1+2) = \frac{9}{6} = \frac{3}{2}$

1 step 1 step

$\mathcal{F}_1\{f\} = \frac{1}{6}\left(1\cdot 1 + 2\cdot\frac{1-i\sqrt{3}}{2} + 1\cdot\frac{-1-i\sqrt{3}}{2} + 2\cdot -1 + 1\cdot\frac{-1+i\sqrt{3}}{2} + 2\,\frac{1+i\sqrt{3}}{2}\right) = 0$

$\mathcal{F}_2\{f\} = \frac{1}{6}\left(1\cdot 1 + 2\cdot\frac{-1-i\sqrt{3}}{2} + 1\cdot\frac{-1+i\sqrt{3}}{2} + 2\cdot 1 + 1\cdot\frac{-1-i\sqrt{3}}{2} + 2\,\frac{-1+i\sqrt{3}}{2}\right) = 0$

2 steps 2 steps

$\mathcal{F}_3\{f\} = \frac{1}{6}\left(1\cdot 1 + 2\cdot -1 + 1\cdot 1 + 2\cdot -1 + 1\cdot 1 + 2\cdot -1\right) = \frac{-3}{6} = -\frac{1}{2}$

3 steps 3 steps

$\mathcal{F}_4\{f\} = \overline{\mathcal{F}_2\{f\}} = 0$

$\mathcal{F}_5\{f\} = \overline{\mathcal{F}_1\{f\}} = 0$

⊛ So: $\mathcal{F}\{f\} = \begin{bmatrix} \frac{3}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}$

6. *(4+16=20pts)* The following parts are about different stages of the fast Fourier transform.

(A) If $\mathbf{f} = \begin{array}{c} \begin{array}{cccc} f_0 & f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \end{array}$, write the even and odd subvectors $\mathbf{f}_{\text{even}}$ and $\mathbf{f}_{\text{odd}}$ used for fast Fourier.

$$f_{even} = \begin{array}{c} \begin{array}{cc} f_0 & f_2 \end{array} \\ \begin{bmatrix} 1 & 3 \end{bmatrix} \end{array}$$

$$f_{odd} = \begin{array}{c} \begin{array}{cc} f_1 & f_3 \end{array} \\ \begin{bmatrix} 2 & 4 \end{bmatrix} \end{array}$$

(B) Suppose that a different vector $\mathbf{g}$ with length 8 has even and odd subvectors with coefficients

$\mathcal{F}_1\{\mathbf{g}_{\text{even}}\} = 2 + 4i$ and

$\mathcal{F}_1\{\mathbf{g}_{\text{odd}}\} = \sqrt{2}\,i.$

$\rightarrow$ use $\overline{\omega_8} = \frac{1-i}{\sqrt{2}} \left(= \frac{\sqrt{2} - i\sqrt{2}}{2}\right)$

Compute all of the coefficients of $\mathcal{F}\{\mathbf{g}\}$ which are possible to find with this information.

$$\mathcal{F}_1\{g\} = \frac{1}{2}\left(\mathcal{F}_1\{g_{even}\} + \overline{\omega_8}\, \mathcal{F}_1\{g_{odd}\}\right) = \frac{1}{2}\left(2+4i + \frac{1-i}{\sqrt{2}}\sqrt{2}\,i\right) = \boxed{\frac{3+5i}{2}}$$

$+\frac{N}{2}$

$$\mathcal{F}_3\{g\} = \overline{\mathcal{F}_5\{g\}} = \boxed{\frac{1-3i}{2}}$$

conjugate

$$\mathcal{F}_5\{g\} = \frac{1}{2}\left(\mathcal{F}_1\{g_{even}\} - \overline{\omega_8}\, \mathcal{F}_1\{g_{odd}\}\right) = \frac{1}{2}\left(2+4i - \frac{1-i}{\sqrt{2}}\sqrt{2}\,i\right) = \boxed{\frac{1+3i}{2}}$$

conjugate

$$\mathcal{F}_7\{g\} = \overline{\mathcal{F}_1\{g\}} = \boxed{\frac{3-5i}{2}}$$

7. *(4pts)* Compute the infinite convolution $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

reverse

$$\begin{array}{rrrrrrrr}
 & & & -1 & 1 & -1 & 1 & \\
 & & & 4 & 3 & 2 & 1 & \times \\
\hline
 & & & -1 & 1 & -1 & 1 & \\
 & & -2 & 2 & -2 & 2 & & \\
 & -3 & 3 & -3 & 3 & & & \\
-4 & 4 & -4 & 4 & & & & + \\
\hline
-4 & 1 & -3 & 2 & 2 & 1 & 1 &
\end{array}$$

reverse

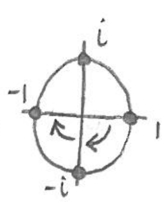
$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -3 \\ 1 \\ -4 \end{bmatrix}$$

8. *(12pts)* Show that $\mathcal{F}_2\{\mathbf{f} \circledast \mathbf{g}\} = 4\mathcal{F}_2\{\mathbf{f}\} \cdot \mathcal{F}_2\{\mathbf{g}\}$ where $\mathbf{f} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

- $f \circledast g = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 2 \end{bmatrix}$

$$\begin{array}{rrr|rrrr|l}
 & & & -1 & 1 & -1 & 1 & \\
 & & & 4 & 3 & 2 & 1 & \times \\
\hline
 & & & -1 & 1 & -1 & 1 & \\
 & & -2 & 2 & -2 & 2 & -2 & \\
 & -3 & 3 & -3 & 3 & -3 & 3 & \\
-4 & 4 & -4 & 4 & -4 & 4 & -4 & + \\
\hline
 & & & 2 & -2 & 2 & -2 &
\end{array}$$

2 steps in



- $\mathcal{F}_2\{\underline{f}\} = \frac{1}{4}\left(1\cdot 1 - 1\cdot(-1) + 1\cdot 1 - 1\cdot(-1)\right) = 1$
- $\mathcal{F}_2\{\underline{g}\} = \frac{1}{4}\left(1\cdot 1 + 2\cdot(-1) + 3\cdot 1 + 4\cdot(-1)\right) = -\frac{1}{2}$
- $\mathcal{F}_2\{\underline{f} \circledast \underline{g}\} = \frac{1}{4}\left(-2\cdot 1 + 2\cdot -1 - 2\cdot 1 + 2\cdot -1\right) = -2$

$\mathcal{F}_2\{\underline{f} \circledast \underline{g}\} \overset{?}{=} 4\mathcal{F}_2\{\underline{f}\}\cdot\mathcal{F}_2\{\underline{g}\}$

$-2 \overset{?}{=} 4 \cdot 1 \cdot -\frac{1}{2}$

✓